Feedback control in coupled map lattices

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(Received 16 January 1998)

Recently, Paramananda *et al.* [Phys. Rev. E **56**, 239 (1997)] have reported the suppression of spatiotemporal chaos using a feedback technique. We present analytic results for the same and compare with numerical results presented in this work. We also suggest an improved method for stabilizing periodic solutions and achieving target cluster states in a spatiotemporal system. [S1063-651X(98)02806-2]

PACS number(s): 05.45.+b

Control of spatiotemporal systems is an important and difficult problem which has attracted attention of late. One of the methods for control was propsed by Ott, Grebogi, and Yorke (OGY) [1]. It requires reasonable knowlege of dynamics and is well suited for low-dimensional systems. The other method is feedback [2,3] (see also [4,5]) which does not need it and is thus suitable for very large dimensional systems. (Engineers have been using this method for quite some time.) In dynamical systems, achieving synchronization is often referred to as control. This happens since quite a few times the synchronized state is indeed a desired state and working out the conditions for synchronization is tantamount to the conditions for control. (Good examples from an applications point of view would be Josephson junction arrays and synchronizing electrical circuits for secure communication.) Apart from that, from a control theory point of view these two problems turn out to be very similar [6]. Coupled map lattices have been a prototypical toy model for spatiotemporal systems. Methods which have been tried in this context include pinning which basically gives a feedback to the system from the desired orbit at periodically spaced sites [7]. Feedback has an appeal in high-dimensional systems due to its simplicity. Of late, Paramananda, Hildebrand, and Eiswirth have studied coupled map lattice systems with feedback and obtained some encouraging numerical results [8]. In this work we will be presenting some analytic calculations along similar lines.

Let us consider a straightforward case of a single map with feedback. The equation is of the type

$$x_{n+1}=f(x_n)+\gamma(x_n-x_{n-1}). \tag{1}$$

This is a two-dimensional system which we can express as $(x_{n+1},x_n)=F(x_n,x_{n-1})=(f(x_n)+\gamma(x_n-x_{n-1}),x_n)$ and one can write the Jacobian at time t as

$$J_t=\begin{pmatrix} f'(x_t)+\gamma & -\gamma \\ 1 & 0 \end{pmatrix}. \tag{2}$$

The eigenvalues of this matrix give the stability constraints. We will see that in a spatially extended system, also, these constraints stay, though some more conditions come in. For the other functional form which is attempted,

$$x_{n+1}=f(x_n)+\gamma(x_n-X_F), \tag{3}$$

where X_F is a stable fixed point $[f(X_F)=X_F]$, it remains a one-dimensional system. But it is easy to see that the stability condition is simply that $|f'(x)+\gamma|<1$. However, it is clear that the periodic orbit solution for a period higher than unity will not be the same as for the $\gamma=0$ case in the case of Eq. (1) or (3). Consequently (as also noted in [6]), in the case of a periodic solution with period p, $X_0,X_1,\ldots,X_{p-1}$, the control does not go to zero asymptotically. A simple solution to this problem would be to drive the system as

$$x_{n+1}=f(x_n)+\gamma(x_n-x_{n-p}) \tag{4}$$

instead of Eq. (1). Similarly, instead of Eq. (3) we can have the evolution rule

$$x_{n+1}=f(x_n)+\gamma(x_n-X_r), \tag{5}$$

where $n=kp+r$, k is an integer, and $0\leqslant r\leqslant p-1$. The stability conditions are $|\Pi_{i=0}^{p-1}f'(X_i)+\gamma|<1$. Of course, one may choose to apply this feedback only rarely; e.g., one may drive the system as

$$x_{n+1}=f(x_n)+\gamma(x_n-X_r)\,\delta_{r,0}. \tag{6}$$

Here the stability condition changes and is given by $|f'(X_0+\gamma)\Pi_{i=1}^{p-1}f'(X_i)|<1$. Thus one may choose to apply perturbations only at those points on the orbit which have strongly expanding eigenvalues. Various states, which are on the same orbit but differ in phase, do not remain equivalent in this kind of driving. Apart from the fact that control goes to zero asymptotically, one more advantage is that the stabilized states are the periodic solutions of the function f and not the altered states. The strategy in Eqs. (5) and (6) can help in getting rid of clustered states (which will be defined later) in coupled map lattices. One more interesting point to be noted is that the above stability conditions do not ensure that if a state were stable without feedback, it would stay stable. (This prompts another possibility of using feedback to increase chaos.)

Now let us consider the coupled map lattice system. Let us consider the local feedback

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{2}[f(x_n(i-1))+f(x_n(i+1))] + \gamma\left(x_n(i)-\frac{1}{N}\sum_{j=1}^{N}x_{n-1}(j)\right). \tag{7}$$

This is a $2N$-dimensional system. We can write $(\mathbf{X}_{t+1},\mathbf{X}_t)=G(\mathbf{X}_t,\mathbf{X}_{t-1})$ where $\mathbf{X}_t=(x_t(1),x_t(2),\ldots,x_t(N))$. The Jacobian J_t at time t is given by

$$\begin{pmatrix} A_t & -\frac{\gamma}{N}\mathbf{1}_N \\ I_N & \mathbf{0}_N \end{pmatrix}, \tag{8}$$

where $\mathbf{0}_N$ is an $N\times N$ matrix with all entries 0. $\mathbf{1}_N$ is an N-dimensional matrix with all entries 1. The matrix A_t is given by

$$\begin{pmatrix} (1-\epsilon)f'(x_t(1))+\gamma & \epsilon/2f'(x_t(2)) & 0 & \dots & \epsilon/2f'(x_t(N)) \\ \epsilon/2f'(x_t(1)) & (1-\epsilon)f'(x_t(2))+\gamma & \epsilon/2f'(x_t(3)) & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \epsilon/2f'(x_t(N)) \\ \epsilon/2f'(x_t(1)) & 0 & 0 & \dots & (1-\epsilon)f'(x_t(N))+\gamma \end{pmatrix}. \tag{9}$$

We assume periodic boundary conditions [10]. It is easy to see that for the homogeneous solution the matrix A_t is a circulant matrix [9,11]. It can be diagonalized by a Fourier matrix F, the entries of which are independent of the matrix being diagonalized. We also note that the matrix $\mathbf{1}_N$ is also circulant and thus diagonalizable by the same matrix. The identity matrix remains unaltered under a similarity transformation under the Fourier matrix and so is the case with $\mathbf{0}_N$. Thus the above matrix has four $N\times N$ blocks, all of which can be diagonalized by the Fourier matrix. Let us define $U_t = G^{-1}J_tG$ where

$$G=\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}. \tag{10}$$

It is easy to see that

$$U=\begin{pmatrix} A_d & B_d \\ I_N & 0 \end{pmatrix}, \tag{11}$$

where each of the blocks is an N-dimensional diagonal matrix. A_d has elements $A_d(l,l)=(1-\epsilon)f'(x_t)+\gamma+\epsilon\cos(\theta_l)f'(x_t)$ where $\theta_l=2\pi l/N$, $l=1,\dots,N$. The matrix B_d has elements $B_d(N,N)=-\gamma$, $B_d(i,i)=0$, $i=1,2,\dots,N-1$. By a simple rearrangement of variables

$$(x_t(1),x_t(2),\dots,x_t(N),x_{t-1}(1),x_{t-1}(2),\dots,x_{t-1}(N)) \\ \to (x_t(1),x_{t-1}(1),x_t(2),x_{t-1}(2),\dots,x_t(N),x_{t-1}(N))$$

the above matrix can be cast in a block diagonal form of N blocks of 2×2 matrices. The required similarity transformation is easy to derive. Thus

$$U_t \equiv \begin{pmatrix} L_1(t) & 0 & \dots & 0 \\ 0 & L_2(t) & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & L_N(t) \end{pmatrix}, \tag{12}$$

where

$$L_i(t)=\begin{pmatrix} A_d(i,i) & B_d(i,i) \\ 1 & 0 \end{pmatrix}. \tag{13}$$

We denote by L_i the blocks corresponding to linearization around the fixed point. The stability conditions will be given by the eigenvalues of each of the blocks. It is interesting that L_N is the stability matrix for a single map. Thus if a single map could not be stabilized by feedback, the lattice cannot be stabilized by feedback. In all other blocks $(i\neq N)$, $B_d(i,i)=0$. Thus the stability conditions are given by the eigenvalues of the matrix A_t. The eigenvalues of A_t are bounded between $(1-2\epsilon)f'(x_t)+\gamma$ and $f'(x_t)+\gamma$. Thus for a fixed point the conditions are straightforward and eigenvalues of a single map with feedback linearized around a fixed point should be within the unit circle in the complex plane and $|(1-2\epsilon)f'(x_F)+\gamma|<1$ and $|f'(x_F)+\gamma|<1$. One can easily check that the value at which stabilization is achieved by Paramananda *et al.* is within the allowed range of $\gamma\in(0.87,\dots,1)$ for $\epsilon=0.08$ and $f(x)=1-1.81x^2$.

Let us try to understand the effect of global feedback. It is defined in [6] as

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{2}[f(x_n(i-1))+f(x_n(i+1))] \\ +\gamma\left(\frac{1}{N}\sum_{j=1}^{N}x_n(j)-\frac{1}{N}\sum_{j=1}^{N}x_{n-1}(j)\right). \tag{14}$$

In this case the analysis can be done in a similar way to the above since the essential symmetries which faciliated the reduction of the problem of diagonalizing the $2N\times 2N$ matrix to N matrices of size 2 are still there. It is easy to check that in this case $A_d(N,N)=f'(x(t))+\gamma$ which is the same as in the case of local feedback. However, $A_d(l,l)=(1-\epsilon)f'(x_t)+\epsilon f'(x_t)\cos(\theta_l)$ $(l=1,\dots,N-1)$ and they do not depend on γ. $B_d(i,i)$'s $(i=1,\dots,N)$ remain unchanged. Thus there is no possibility of stabilizing the unstable fixed point with this feedback since one of the eigenvalues (in the large-N limit) will approach $f'(x_F)$ for the synchronized fixed point. Thus it is not a surprise that Paramannda *et al.* are not able to stabilize the homogeous state starting with arbitrary initial conditions. However, they do the following. They reset the parameter to get a system in a synchronized state. Later they reset the parameter back to the old value and add control and say that applying control leads to a synchronized state. The reason why this works is that the evolution equations are such that if you start the system in a synchronized state, the system will stay synchronized. Now the question is why it goes to a fixed point which was previously unstable. The mode under consideration now is only the homogeneous mode. Stability of this mode essentially means that a single map can be stabilized by feedback. Thus if an unstable fixed point of a single map can be stabilized by feedback, one could go to a homogeneous fixed point state starting from the synchronized initial condition. However, one should note that starting with a small disturbance around the homogeneous state it will not be possible to stabilize the system to a homogeneous unstable fixed point. Since a single system can be stabilized with feedback for $\gamma \in (0.43, \ldots ,1)$ (for $a=1.81$), one can observe a homogeneous unstable fixed point in this regime with synchronized initial conditions which is possible in numerical simulations.

Now an interesting observation can be made. This could be done because the Nth block has a very different structure from the rest of the blocks. This gives a possibility that one of the eigenvalues can be well separated from the rest. Thus it may even be possible to have synchronous chaos in systems with feedback if one starts with a synchronized state.

The above analysis for a synchronized state can easily be extended to the case when feedback is from a state p time steps before the current time step. Let the evolution be of the type

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{2}[f(x_n(i-1)+f(x_n(i+1))] + \gamma\left(x_n(i)-\frac{1}{N}\sum_{j=1}^{N} x_{n-p}(j)\right). \quad (15)$$

The stability conditions for a synchronized state can be simply derived using a similar technique as above. We will write the final result. The Jacobian in the above case is the $pN\times pN$ matrix which can be reduced to N matrices of size $p\times p$ in block diagonal form as Eq. (12) except that each of the blocks now is a $p\times p$ matrix given by

$$L_i(t)=\begin{pmatrix} A_d(i,i) & 0 & \ldots & 0 & B_d(i,i) \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \quad (16)$$

Linearization is around the homogeneous state at time t. Thus the stability conditions will be given by the eigenvalues of the N product matrices given by $L_i=\Pi_{i=1}^{p}L_i(t)$. This can be done since the above form is obtained by similarity transformations which do not depend on individual matrices but their symmetries. It is interesting that L_N is the stability matrix for a single map. Thus if a single map could not be stabilized by feedback, the lattice cannot be stabilized by the feedback. The periods stabilized could be p and its harmonics. For example, applying feedback after ten iterates may result in periods 1, 2, 5, or 10 if the control signal goes to zero. If a fixed point, i.e., period unity, is stabilized, then the conditions are very simple. In all the blocks other than L_N, $B_d(i,i)=0$. Thus the stability conditions are given by the eigenvalues of the matrix A_d. The eigenvalues of A_d at time t are bounded between $(1-2\epsilon)f'(x_t)+\gamma$ and $f'(x_t)+\gamma$. Thus the conditions are that eigenvalues of a single map with feedback linearized around fixed point should be within the unit circle in the complex plane and that $|(1-2\epsilon)f'(x_F)+\gamma|<1$ and $|f'(x_F)+\gamma|<1$. For higher periods, the stability condition will be that eigenvalues of matrices $\Pi_{t=1}^{p}L_i(t)$ be in the unit circle for $i=1,\ldots ,N$ as explained above. Since $B_d(i,i)=0$ for $i\neq N$, the eigenvalues are just a product of $A_d(i,i)$'s at successive times for blocks other than the Nth block.

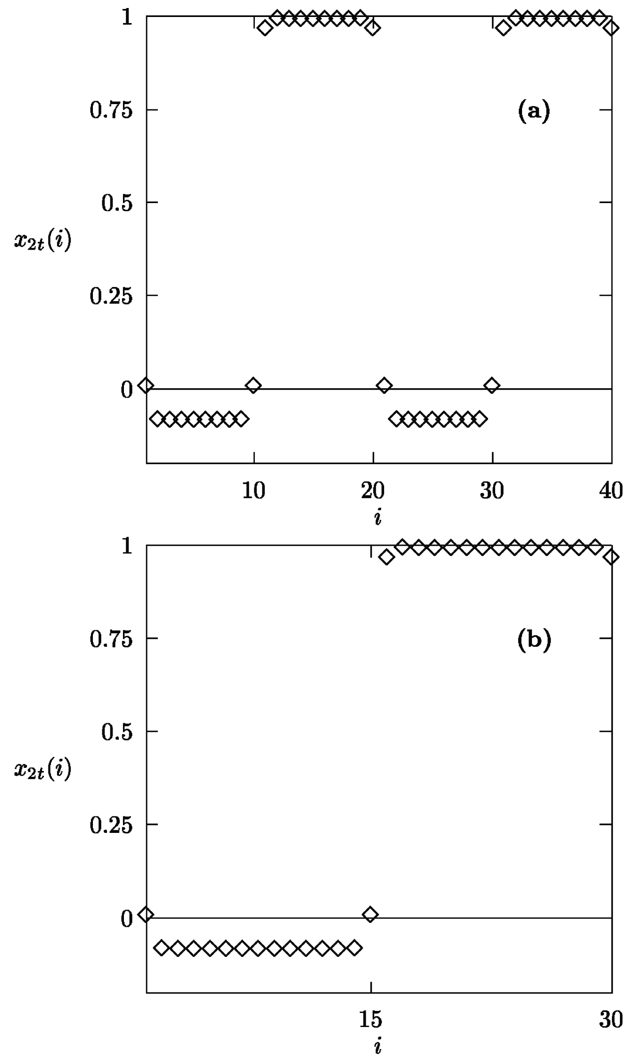

FIG. 1. (a) The superposition of the asymptotic state of 20 random initial conditions controlled to the (10(0),10(1),10(0),10(1)) state. (b) Similar figure for control to the (15(0),15(1)) state.

In the Ref. [6], other structures apart from the homogeneous state which are observed are clustered attractors. These are unsynchronized periodic attractors with period other than unity. There are two points to be made. (a) The clustered state depends entirely on the initial conditions and with a simple evolution rule as above one cannot target it to a desired clustered state. (b) Control does not go to zero asymptotically.

Now if one wants to have a control mechanism, one should be able to suggest a way in which a desired clustered state can be achieved starting with random initial conditions and also a mechanism to switch between different cluster states. The physical origin of the clustered states is easy to see. In the case of a single map, the dynamical evolution leads to states periodic with period p, say, $X_0, X_1, \ldots, X_{p-1}$. For zero coupling p^N states are asymptotically possible for an assembly of N maps (actually, p^{N-1} after accounting for phase degeneracy). Of course, for a nonzero coupling, all p^N states may neither exist nor be stable. But still it is reasonable to argue that a large number of accesible states is possible. Thus unless the feedback is with a certain phase and breaks the symmetry, the systems will not get synchronized. Now if one is targeting to get the clustered states and one knows the local dynamics, a strategy similar to Eq. (5) can be tried. We drive sites in different clusters with different phase but within the cluster the phase is maintained. One expects a desired cluster state to be achieved this way if it exists and is stable (with feedback). Let us denote $\mathcal{C}=(k_1(p_1),k_2(p_2),\ldots,k_n(p_n))$ as a cluster state. where $k_1+k_2+\cdots+k_n=N$ and $0\leqslant p_i\leqslant p-1$ for $i=1,\ldots,n$. This is a state in which first k_1 sites are near X_{p_1}, next k_2 sites are near X_{p_2} (within some tolerence), and so on. Now we can define the following evolution rule:

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{2}[f(x_n(i-1))+f(x_n(i+1))] + \gamma(x_n(i)-X_{(\mathrm{mod}(n+p1,p))}) \tag{17}$$

for $1\leqslant i\leqslant k_1$,

$$x_{n+1}(i)=(1-\epsilon)f(x_n(i))+\frac{\epsilon}{2}[f(x_n(i-1)+f(x_n(i+1))] + \gamma(x_n(i)-X_{(\mathrm{mod}(n+p2,p))}) \tag{18}$$

for $1\leqslant i\leqslant k_2$, and so on. We were able to stabilize period-2 clusters by this approach. For $a=1.1$ of the logistic map $f(x)=1-ax^2$ we could go to a target state of $\mathcal{C}=(15(0),15(1))$; i.e., $i=1,\ldots,k_1=15$ are in one cluster and $i=k_1+1,\ldots,k_2=30$ are in the other cluster. We applied $\gamma=0.1$, $\epsilon=0.08$. Similarly we could stabilize the $(10(0),10(1),10(0),10(1))$ cluster for similar conditions. Of course, with high periods and states very near each other targeting to a desired state is difficult. Figure 1(a) shows a superposition of asymptotic states of 20 random initial conditions targeted to the $(10(0),10(1),10(0),10(1))$ state. Figure 1(b) shows the superposition of 20 random initial conditions targeted to the $(15(0),15(1))$ state.

The procedure stated above can obviously be used for coupled Henon maps or coupled Lozi maps or coupled area-preserving maps which have an evolution of the type $x_{t+1}=F(x_t,x_{t-1})$. The whole calculation also goes over to coupled oscillators (with inertia). The above calculation is for a synchronized case. For a spatially periodic case, a lengthy but straightforward calculation on the lines of this paper and Ref. [9] is needed and this work is in progress. The calculation on lines of [9] would be useful for analyzing the effect of feedback at periodically spaced sites in coupled maps [5]. However, a more pertinent question would be as to how much intuition carries over to the continuous time dynamical systems with feedback. The answer is not clear. We feel that a reasonable part goes over. For feedback in the single oscillator case the analysis is strikingly similar to the case of a map [12]. Of course, details change. For example, one does not need the orbit of a discrete system to have finite torsion to be stabilized by feedback. Analysis for the coupled oscillator case has not been carried out to our knowledge.

This work was supported in parts by grants from the Hong Kong Research Grants Council (RGC) and the Hong Kong Baptist University. The author acknowledges the theory group MRC (IISc) for allowing the use of their computational facilities.

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